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Quantum and classical solutions for an electron–dyon system

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Abstract. Classical solutions for a relativistic electron–dyon system are found in terms of the action variables. A formula for the energy of the system is obtained and comparison with the corresponding quantum mechanical formula is made, with results showing that classical physics gives the correct single-particle energy formula not including quantum corrections, which in turn are essential to the issue of breakdown.

1. Introduction

Recent work on superconducting cosmic strings has renewed the interest in the problem of the stability of the QED vacuum in a strong inhomogeneous magnetic field [1]. Amsterdamski [2] supposed that pairs are produced even by a DC current in a straight string, while Gornicki *et al* [3] indicated that there is no pair creation in an arbitrary static magnetic field. So the issue remains to be investigated. In the case of a strong inhomogeneous electric field, the situation is simple. By studying energy levels of the electron–nucleus system, the conclusion is drawn for an extended nucleus that with the increase of the nucleus charge Z the bound state energy level lowers, and eventually at the critical value Z_c, it begins to 'dive' into the negative energy continuum, and e^+e^- pairs are created [4]. A similar reasoning seems to be applicable to the case of a magnetic field. One may consider the bound state of an electron–monopole system and find the critical monopole charge, if it exists. But, as is well known, there are some mathematical difficulties which handicap such an approach [5]. Instead we consider an electron–dyon system. A dyon is a Dirac monopole carrying an electric charge which makes a bound state possible. With this model and using a technique very similar to that used formerly for an electron–monopole system, S K Bose [6] solved the corresponding Dirac equation and found a formula for its energy spectrum, which indicates that any non-zero magnetic charge may induce a breakdown only in the presence of a non-zero electric charge, however small it is. Obviously, from Bose's solution we can draw the conclusion that a static inhomogeneous magnetic field in general (considered as the superposition of fields of magnetic charges) may induce pair creation only in the presence of an electric charge.

In view of the fact that Bose's derivation is somewhat difficult to follow and its physical implication is not always clear, it seems profitable to investigate its classical counterpart. In this article we find the classical solution of a relativistic electron–dyon system and obtain a formula for its energy in terms of the action variables, which has a form very similar to that of Bose's, and hence we are able to give a classical explanation for some of Bose's results [7]. This leads to the conclusion that the 'breakdown' for an electron–dyon system is essentially a relativistic effect, and similar situations occur whether classical or quantum theory is invoked.

Throughout this paper, only point-like dyons are considered, generalization to extended dyons is not difficult. In section 2, the classical formula for the energy of an electron–dyon system is obtained; In section 3, a comparison with the corresponding quantum mechanical formula found by Bose is made; the conclusion is drawn that, in both cases, with any non-zero magnetic charge and simultaneously any non-zero electric charge, breakdown occurs. Lastly, in section 4, some general remarks are made.

2. Classical solution for a relativistic electron–dyon system

We treat the relativistic electron–dyon system in action-angle variables, using the standard Hamiltonian formulation [8]. The Hamilton–Jacobi equation for this system may be readily written as

$$\left(\nabla S-\frac{e}{c}\mathbf{A}\right)^2-\frac{1}{c^2}\left(\frac{\partial S}{\partial t}+\frac{Ze^2}{r}\right)^2+m^2c^2=0 \tag{1}$$

where $S(q, \alpha, t)$ is Hamilton's principal function and $\mathbf{A}$ is the vector potential due to the magnetic charge of the dyon; two choices for $\mathbf{A}$ are necessary to avoid the so-called strings of singularities [9]. In polar coordinates one possible choice for $\mathbf{A}$ is

$$A_\theta^{\mathrm{I}}=A_r^{\mathrm{I}}=0 \qquad A_\phi^{\mathrm{I}}=\frac{g}{r\sin\theta}(1-\cos\theta) \tag{2}$$

which is singular only at $\theta=\pi$, and another one is

$$A_\theta^{\mathrm{II}}=A_r^{\mathrm{II}}=0 \qquad A_\phi^{\mathrm{II}}=-\frac{g}{r\sin\theta}(1+\cos\theta) \tag{3}$$

which is singular only at $\theta=0$. In these formulas g is the magnetic charge.

Hence, in polar coordinates, the Hamilton–Jacobi equation has the form

$$\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial S}{\partial\phi}-\frac{e}{c}A_\phi^{\mathrm{I,II}}\right)^2-\frac{1}{c^2}\left(\frac{\partial S}{\partial t}+\frac{Ze^2}{r}\right)^2+m^2c^2=0. \tag{4}$$

Making the transformation from $S(q, \alpha, t)$ to Hamilton's characteristic function $W(q, \alpha)$ and separating the variables, we obtain

$$\left(\frac{\partial W_r}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial W_\theta}{\partial\theta}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial W_\phi}{\partial\phi}-\frac{e}{c}A_\phi^{\mathrm{I,II}}\right)^2-\frac{1}{c^2}\left(-E+\frac{Ze^2}{r}\right)^2+m^2c^2=0 \tag{5}$$

where W_r, W_θ and W_ϕ are functions of r, θ and ϕ, respectively, $W=W_r(r)+W_\theta(\theta)+W_\phi(\phi)$. Equation (5) can be solved in a routine way, but we are primarily interested in obtaining the action variables.

Setting

$$\frac{\partial W'_\phi}{\partial \phi} = \alpha_\phi \tag{6}$$

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\alpha_\phi - \frac{er\sin\theta}{c}A_\phi^{\mathrm{I,II}}\right)^2 = \alpha_\theta^2 \tag{7}$$

the Hamilton–Jacobi equation reduces to

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} = \frac{1}{c^2}\left(-E + \frac{Ze^2}{r}\right)^2 - m^2c^2. \tag{8}$$

The action variables J_ϕ, J_θ and J_r are determined by the following definition.

$$J_i = \oint \frac{\partial W}{\partial x_i}\,\mathrm{d}x_i \qquad x_i = r, \theta, \phi. \tag{9}$$

Their explicit expressions may be obtained by using equations (6)–(8):

$$J_\phi = \oint \alpha_\phi\,\mathrm{d}\phi \tag{10}$$

$$J = \oint \left\{\alpha_\theta^2 - \frac{1}{\sin^2\theta}\left(\alpha_\phi - \frac{er\sin\theta}{c}A_\phi^{\mathrm{I,II}}\right)^2\right\}^{1/2}\mathrm{d}\theta \tag{11}$$

$$J_r = \oint \left\{\frac{1}{c^2}\left(-E + \frac{Ze^2}{r}\right)^2 - m^2c^2 - \frac{\alpha_\theta^2}{r^2}\right\}^{1/2}\mathrm{d}r. \tag{12}$$

The first integral immediately yields the result $J_\phi = 2\pi\alpha_\phi$. The second integral is a little tedious; to evaluate it we first change the variable θ to z

$$\mathrm{e}^{\mathrm{i}\theta} = z \qquad \cos\theta = \tfrac{1}{2}(z + z^{-1}) \qquad \sin\theta = \frac{1}{2\mathrm{i}}(\mathrm{z} - \mathrm{z}^{-1}) \tag{13}$$

and the integration contour is a unit circle whose centre is the origin. The resultant integration may be performed by using the method of residues [10]. Care should be taken that the integrand has three poles: a pole at $z=0$ inside the contour and two poles $z=\pm 1$ on the contour. To find the residue of the integrand relative to the pole $z=0$, we expand the integrand in a Taylor series about $z=0$ and evaluate the residue easily

$$R_0 = -\mathrm{i}\left\{\alpha_\theta^2 + \left(\frac{eg}{c}\right)^2\right\}^{1/2} \tag{14}$$

whether A_ϕ^{I} or A_ϕ^{II} is used. As to the residues relative to $z=\pm 1$, the calculation is a little involved. For $z=+1$, A_ϕ^{I} should be used and expansion about $z=+1$ furnishes the residue

$$R_{+1} = \mathrm{i}\alpha_\phi \tag{15}$$

while for $z=-1$, A_ϕ^{II} should be used and the residue is also

$$R_{-1} = \mathrm{i}\alpha_\phi. \tag{16}$$

Taking all these into account and using the residue theorem, we obtain

$$\begin{aligned} J_\theta &= 2\pi\left\{\alpha_\theta^2+\left(\frac{eg}{c}\right)^2\right\}^{1/2}-2\pi\alpha_\phi \\ &= 2\pi\left\{\alpha_\theta^2+\left(\frac{eg}{c}\right)^2\right\}^{1/2}-J_\phi. \end{aligned} \tag{17}$$

Solving for α_θ^2 and substituting the result into (12), we obtain

$$J_r=\oint\left\{\frac{E^2}{c^2}-m^2c^2-\frac{1}{c^2}\,\frac{2EZe^2}{r}-\frac{1/4\pi^2(J_\theta+J_\phi)^2-(eg/c)^2-(Ze^2/c)^2}{r^2}\right\}^{1/2}\mathrm{d}r. \tag{18}$$

This contour integral can also be evaluated by using the method of residues [8] with the result

$$J_r=-\left\{(J_\theta+J_\phi)^2-4\pi^2\left(\frac{Ze^2}{c}\right)^2-4\pi^2\left(\frac{eg}{c}\right)^2\right\}^{1/2}-\frac{2\pi EZe^2}{c(-E^2+m^2c^4)^{1/2}}. \tag{19}$$

Solving for E, we obtain

$$\frac{E}{mc^2}=\left\{1+\left[\frac{2\pi Ze^2}{J_rc+[(J_\theta+J_\phi)^2c^2-(2\pi Ze^2)^2-(2\pi eg)^2]^{1/2}}\right]^2\right\}^{-1/2}. \tag{20}$$

This is our main result. Two special cases are of particular interest. For $g=0$, i.e. with no magnetic charge, the above equation reduces to the classical expression for the energy of a relativistic electron-nucleus system. On the other hand, for $Z=0$, i.e. only the monopole is considered, $E=mc^2$, no bound states exist. Note that here and there we suppose no magnetic moment adherent to the electron [5].

3. Comparison with relativistic quantum theory

Bose [6] solved the Dirac equation for an electron–dyon system and found the formula for its energy spectrum as follows

$$\frac{E}{mc^2}=\left\{1+\left[\frac{Z\alpha}{n_r+[(J+\frac{1}{2})^2-(Z\alpha)^2-(eg/\hbar c)^2]^{1/2}}\right]^2\right\}^{-\frac{1}{2}} \tag{21}$$

where $\alpha=e^2/\hbar c$ is the fine structure constant. Formula (20) is formally very similar to formula (21). According to Bohr–Sommerfeld quantum conditions, action variables $J_i=\oint P_i\,\mathrm{d}q_i$ should be integers multiplied by Planck's constant; the two formulas are nearly identical except for a half integer quantum number which, as is well known, distinguishes quantum mechanics from the old quantum theory.

The implications of these two formulas as to the role of magnetic charge or/and electric charge in the occurrence of breakdown are also similar.

For a simple electric charge, $g=0$, equations (20) and (21) reduce to, respectively

$$\frac{E}{mc^2}=\left\{1+\left[\frac{2\pi Ze^2}{J_rc+[(J_\theta+J_\phi)^2c^2-(2\pi Ze^2)^2]^{1/2}}\right]^2\right\}^{-1/2} \tag{22}$$

and

$$\frac{E}{mc^2}=\left\{1+\left[\frac{Z\alpha}{n_r+[(J+\frac{1}{2})^2-(Z\alpha)^2]^{1/2}}\right]^2\right\}^{-1/2}. \tag{23}$$

In the classical formula (22), by making the angular momentum small enough one makes negative the argument of the square root in the denominator of the second term in parentheses, so the classical formula (22) gives a breakdown for any nuclear charge Z, while quantum mechanics puts a lower bound on the angular momentum of the electron–nucleus system and hence yields a threshold value 137 for a point electric charge Z in (23). On the other hand, for a given angular momentum L, we can define a 'strong' Coulomb potential as one for which $Ze/Lc>1$, i.e. which causes the radius of circular orbits to shrink to zero, with this, Garcia [7] gave a classical analogue of the breakdown of the Dirac hydrogenic atom equations.

If there is a magnetic charge simultaneously, $g\neq 0$, the classical formula (20) shows that for any non-zero electric charge, by making the angular momentum sufficiently small, breakdown occurs; while in quantum formula (21), the orbital angular momentum acquires a minimum value equal to the product of electric and magnetic charges in natural units, thus changing the threshold value from $Z=137$ to $Z=0$. So in both cases, with any non-zero magnetic charge and simultaneously any non-zero electric charge, breakdown, being essentially a relativistic effect, occurs.

4. Remarks

In conclusion some general remarks are made:

(1) For the energy of an electron–dyon system both our classical relativistic formula and Bose's relativistic quantum formula lead to the same conclusion: a magnetic charge may induce a breakdown only in the presence of an electric charge, however small it is. This conclusion is helpful to settle the issue pointed out in the beginning of this article [11].

(2) All our discussions are limited to pointlike dyons, in reality the dyon is extended; to generalize our results to an extended dyon is not difficult, and we may proceed in perfect analogy to the case of an electron–nucleus system [12], as noted earlier.

(3) If Witten's effect [13] is taken into account, a magnetic pole may induce an electric charge, then on the basis of the above discussion, an electron–monopole system may behave in a similar way to an electron–dyon system.

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